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INTERACTION OF A DIE WITH A LAYERED ELASTIC FOUNDATION

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Plane and axisymmetric contact problems for a three-layer elastic half-space are considered. The plane problem is reduced to a singular integral equation of the first kind whose approximate solution is obtained by a modified Multhopp–Kalandiya method of collocation. The axisymmetric problem is reduced to an integral Fredholm equation of the second kind whose approximate solution is obtained by a specially developed method of collocation over the nodes of the Legendre polynomial. An axisymmetric contact problem for an transversely isotropic layer completely adherent to an elastic isotropic half-space is also considered. Examples of calculating the characteristic integral quantities are given.

Key words: *contact problem, layered elastic foundation, die.*

1. Auxiliary Problem. We consider the problem of equilibrium of an elastic half-plane with a two-layer elastic coating (Fig. 1) under conditions of plane deformation. Complete adhesion is reached between the layers $H - h \le y \le H$ and $0 \le y \le h$, and also between the lower layer $0 \le y \le h$ and the half-plane $y \le 0$. The upper layer is loaded by a distributed normal pressure $q(x)$. The mechanical parameters (shear moduli and Poisson's ratios) of the layers and the half-plane are G_j and ν_j $(j = 1, 2, 3)$, respectively.

The boundary condition of the problem have the following form:

— for $y = H$,

$$\sigma_y^{(1)} = -\tilde{q}(x), \qquad \tau_{xy}^{(1)} = 0,$$
$$\tilde{q}(x) = q(x), \quad |x| \le a, \qquad \tilde{q}(x) = 0, \quad |x| > a; \tag{1.1}$$

— for $y = h$,

$$u_1 = u_2, \qquad v_1 = v_2, \qquad \sigma_y^{(1)} = \sigma_y^{(2)}, \qquad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}; \tag{1.2}$$

— for $y = 0$,

$$u_2 = u_3, \qquad v_2 = v_3, \qquad \sigma_y^{(2)} = \sigma_y^{(3)}, \qquad \tau_{xy}^{(2)} = \tau_{xy}^{(3)}. \tag{1.3}$$

Here u and v are the displacements along the x and y axes, respectively, σ_y is the normal stress, and τ_{xy} is the shear stress. The boundary conditions (1.1)–(1.3) should be supplemented by the absence of stresses in the structure as $|x| \to \infty$ and $|y| \to \infty$.

To solve the problem, we use the general representation of the solution of the Lamé equations via the biharmonic function of displacements χ:

$$u = -\frac{\partial^2 \chi}{\partial x \, \partial y}, \qquad v = \Big[2(1 - \nu)\Delta - \frac{\partial^2}{\partial y^2}\Big]\chi \tag{1.4}$$

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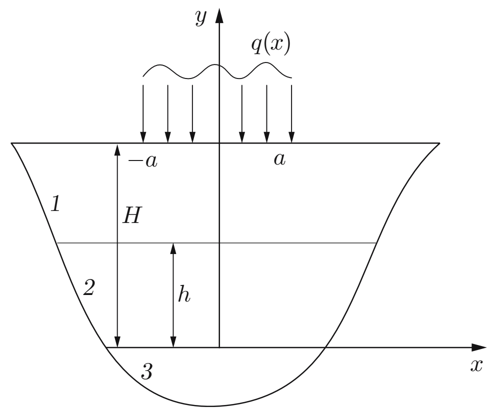

Fig. 1. Geometry of the problem.

(Δ is the Laplace operator). Presenting the biharmonic functions χ_j $(j = 1, 2, 3)$ in (1.4) in the form of the Fourier integrals

$$\chi_j(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_j(\alpha, y)\, e^{-i\alpha x}\, d\alpha, \tag{1.5}$$

we obtain the expressions for the transforms $X_j(\alpha, y)$ in Eq. (1.5)

$$\begin{aligned} X_1(\alpha, y) &= [a_1(\alpha) + |\alpha| y a_2(\alpha)]\, e^{|\alpha| y} + [b_1(\alpha) + |\alpha| y b_2(\alpha)]\, e^{-|\alpha| y}, \\ X_2(\alpha, y) &= [c_1(\alpha) + |\alpha| y c_2(\alpha)]\, e^{|\alpha| y} + [d_1(\alpha) + |\alpha| y d_2(\alpha)]\, e^{-|\alpha| y}, \\ X_3(\alpha, y) &= [e_1(\alpha) + |\alpha| y e_2(\alpha)]\, e^{|\alpha| y}, \end{aligned} \tag{1.6}$$

where ten functions a_l, b_l, c_l, d_l, and e_l $(l = 1, 2)$ have to be found from the boundary conditions (1.1)–(1.3). To find these functions, we express the boundary conditions (1.1)–(1.3) (using the Cauchy formulas relating displacements and strains and Hooke's law relating strains and stresses) in terms of the functions χ_j $(j = 1, 2, 3)$. After that, we present the discontinuous function $\tilde{q}(x)$ of the form (1.1) as the Fourier integral

$$\tilde{q}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\alpha)\, e^{-i\alpha x}\, d\alpha \tag{1.7}$$

and write the boundary conditions (1.1)–(1.3) in Fourier transforms. Using now Eqs. (1.6), we obtain a system of ten algebraic equations for determining the functions a_l, b_l, c_l, d_l, and e_l $(l = 1, 2)$. Let us solve this system.

Substituting the found values of $a_l(\alpha)$ and $b_l(\alpha)$ $(l = 1, 2)$ into the first formula in (1.6), we use the second formula in (1.4) to construct an expression for $v'_x(x, H)$ necessary further to formulate the contact problem:

$$v'_x(x, H) = \frac{i}{2\pi\theta_1} \int_{-\infty}^{\infty} \operatorname{sgn}(\alpha)\, L(|\alpha| H) Q(\alpha)\, e^{-i\alpha x}\, d\alpha, \qquad \theta_1 = \frac{G_1}{1 - \nu_1}. \tag{1.8}$$

The expression for the function $L(u)$ is rather cumbersome and is not given here. We only note that the function $L(u)$ is continuous and possesses the following asymptotic properties:

$$\begin{aligned} L(u) &= 1 + O(e^{-2|u|\delta}), \qquad |u| \to \infty, \quad \delta = \inf(\varepsilon, 1 - \varepsilon), \\ L(u) &= n + O(|u|), \qquad |u| \to 0. \end{aligned} \tag{1.9}$$

2. Plane Contact Problem. We replace the first boundary condition in (1.1) by the following conditions:

$$v'_x(x, H) = -[\beta - f'(x)], \quad |x| \le a, \qquad \sigma_y^{(1)}(x, H) = 0, \quad |x| > a. \tag{2.1}$$

The first boundary condition in (2.1) is the condition of contact between the stiff die whose foundation is described by the function $y = f(x)$ and the surface of the two-layer coating of the elastic half-plane. The second boundary condition in (2.1) implies the absence of the normal load onto the surface outside of the contact region $|x| \le a$. We assume that the die is pressed without friction into a three-layer foundation by a force P that displaces the die by a certain small distance translationally in the negative direction of the y axis and turns the die by a small angle β.

The remaining boundary conditions in (1.1)–(1.3) and the condition of the absence of stresses in the structure at infinity remain unchanged. Then, with allowance for Eq. (1.8), the first boundary condition in (2.1) is satisfied if the following relation is valid:

$$\int_{-\infty}^{\infty} \operatorname{sgn}(\alpha)\, L(|\alpha| H) Q(\alpha)\, \mathrm{e}^{-i\alpha x}\, d\alpha = 2\pi i \theta_1 [\beta - f'(x)], \qquad |x| \le a. \tag{2.2}$$

The second boundary condition (2.1) is satisfied if the inverse transform of Eq. (1.7) is written as

$$Q(\alpha) = \int_{-a}^{a} q(\xi)\, \mathrm{e}^{i\alpha\xi}\, d\xi, \tag{2.3}$$

where $q(x)$ is an unknown contact pressure.

Substituting Eq. (2.3) into Eq. (2.2) and making some simple transformations, we obtain the following integral equation for determining $q(x)$:

$$\int_{-a}^{a} q(\xi)\, d\xi \int_{0}^{\infty} L(u) \sin\Big(u\, \frac{\xi - x}{H}\Big)\, du = \pi \theta_1 H [\beta - f'(x)], \qquad |x| \le a. \tag{2.4}$$

This equation is supplemented by the condition of die equilibrium

$$P = \int_{-a}^{a} q(\xi)\, d\xi, \qquad Pe = \int_{-a}^{a} \xi q(\xi)\, d\xi, \tag{2.5}$$

where e is the distance from the y axis to the line where the force P acts. The second condition in (2.5) serves to determine the die-turning angle β. If the half-length of the contact line a is not defined by the die angle, Eq. (2.4) should be supplemented by the conditions

$$q(\pm a) = 0, \tag{2.6}$$

necessary to find the quantities a and e for a prescribed force P.

3. Method of Solving the Plane Problem. Using the integral

$$\int_{0}^{\infty} \sin(uz)\, du = \frac{1}{z},$$

understood in the general sense, we write the integral equation (2.4) in the following form:

$$\int_{-a}^{a} q(\xi)\, \frac{d\xi}{\xi - x} = \pi \theta_1 [\beta - f'(x)] + \frac{1}{H} \int_{-a}^{a} q(\xi) G\Big(\frac{\xi - x}{H}\Big)\, d\xi,$$

$$G(z) = \int_{0}^{\infty} [1 - L(u)] \sin(uz)\, du. \tag{3.1}$$

Note that the left side of Eq. (3.1) contains a singular operator with the Cauchy kernel, and the right side contains a regular operator, because the function $G(t)$ is continuous, which follows from Eqs. (1.9).

To construct an approximate solution of the integral equation (3.1), we intend to use the modified Multhopp–Kalandiya method [1, 2]. Let us briefly describe its scheme. It can be shown [3] that the general solution of Eq. (3.1) has the form

$$q(x) = \omega(x)/\sqrt{a^2 - x^2}. \tag{3.2}$$

We substitute Eq. (3.2) into Eq. (3.1) and pass to new variables $\xi = a\cos\tau$ and $x = a\cos t$. As a result, we obtain

$$\int_0^\pi \frac{\Omega(\tau)\,d\tau}{\cos\tau - \cos t} = \pi g(t) + \frac{1}{\lambda}\int_0^\pi \Omega(\tau) G\Big(\frac{\cos\tau - \cos t}{\lambda}\Big)\,d\tau,$$
$$\Omega(t) = \frac{\omega(a\cos t)}{a\theta_1}, \qquad g(t) = \beta - f'(a\cos t), \qquad \lambda = \frac{H}{a}. \tag{3.3}$$

For the function $\omega(x)$, we introduce into consideration the Lagrange interpolation polynomial over the nodes

$$x_l = a\cos t_l, \qquad t_l = \pi(2l-1)/(2N), \qquad l = 1, 2, \ldots, N,$$

which are zeros of the Chebyshev polynomial of the first kind $T_N(x/a)$. In particular cases, where $\omega(x)$ is an odd or an even function and $N = 2r+1$ ($r \ge 1$), such polynomials have the form

$$\Omega(t) \simeq \frac{1}{r+1/2}\sum_{l=1}^{r+1}\Omega(t_l)\delta_l\Big(1 + 2\sum_{m=1}^{r}\cos(2mt_l)\cos(2mt)\Big),$$
$$\Omega(t) \simeq \frac{2}{r+1/2}\sum_{l=1}^{r}\Omega(t_l)\Big(\sum_{m=1}^{r}\cos((2m-1)t_l)\cos((2m-1)t)\Big), \tag{3.4}$$

where $\delta_l = 1$ for $l \ne r+1$ and $\delta_l = 1/2$ for $l = r+1$.

Substituting the approximate expressions for $\Omega(t)$ in one of the forms (3.4) into Eq. (3.3) and using expression 7.344(1) from [4]

$$\int_0^\pi \frac{\cos(j\tau)\,d\tau}{\cos\tau - \cos t} = \pi\frac{\sin(jt)}{\sin t}, \qquad 0 \le t \le \pi, \quad j = 0, 1, \ldots,$$

we can exactly calculate the integral in the left side of Eq. (3.3). To approximately calculate the integral in the right side of this equation, we use the Gaussian quadrature

$$\int_0^\pi p(\tau)\,d\tau = \frac{\pi}{N}\sum_{l=1}^{N} p(t_l).$$

Calculating the integrals in (3.3), we assume that $t = t_s$ in the resultant expression and obtain a system of r linear algebraic equations with respect to $\Omega(t_l)$:

$$-\sum_{l=1}^{r+1-p}\Omega(t_l)\delta_l\Big\{\frac{1}{\sin t_s}\chi_r^{(p)}(t_l, t_s) + \frac{1}{2\lambda}\Big[G\Big(\frac{\cos t_l - \cos t_s}{\lambda}\Big)$$
$$-(-1)^p G\Big(\frac{\cos t_l + \cos t_s}{\lambda}\Big)\Big]\Big\} = \Big(r + \frac{1}{2}\Big)g(t_s), \quad s = 1, 2, \ldots, r, \tag{3.5}$$
$$\chi_r^{(p)}(\tau, t) = -2\sum_{m=1}^{r}\cos((2m-p)\tau)\sin((2m-p)t)$$

[$p = 0$ and $p = 1$ for the even and odd function $\omega(x)$, respectively].

To close system (3.5) for the even function $\omega(x)$, we have to add an equation obtained from the first condition in (2.5) with the use of Eq. (3.2) and the first formula in (3.4):

$$\frac{P}{a\theta_1} = \frac{\pi}{r+1/2}\sum_{l=1}^{r+1}\Omega(t_l)\delta_l. \tag{3.6}$$

When system (3.5), (3.6) is solved for the even function $\omega(x)$ and system (3.5) is solved for the odd function $\omega(x)$ with respect to $\Omega(t_l)$, Eqs. (3.4) can be used to find the approximate expressions for the functions $\Omega(t)$ and, hence, the functions $\omega(x)$ and $q(x)$. Further we can use, if necessary, the second condition in (2.5) and conditions (2.6) to determine the quantities β, a, and e.

TABLE 1

λ	c_0	λ	c_0
1/4	2.72	2	1.93
1/2	2.48	4	1.65
1	2.11		

Let us consider an example of a parabolic die with smooth edges under the action of a centrally applied force. The function of the die-foundation shape has the form $f(x) = x^2/(2R)$, where R is the radius of curvature at the apex of the parabola. Using the values of the parameters

$$G_2 = 3G_1/2, \quad G_3 = 2G_1, \quad \nu_1 = 0.25, \quad \nu_2 = 0.35, \quad \nu_3 = 1/3, \quad H/h = 2,$$

we find the dependence of the coupling coefficient between $P/(\theta_1 a)$ and a/R on λ. The values of $c_0 = PR/(\theta_1 a^2)$ calculated for different values of the parameter λ are listed in Table 1.

4. Axisymmetric Contact Problem. We consider an axisymmetric contact problem of pressing a rigid die into a three-layer foundation consisting of two elastic layers that rest on an elastic half-space. The layers are completely adherent to each other and to the half-space. The thicknesses of the upper and lower layers are $H - h$ and h, respectively. According to [5], this contact problem can be reduced to an integral equation of the first kind with respect to the contact pressure $q(r)$ with a symmetric kernel by means of Hankel's transform:

$$\int\limits_0^a q(\rho)K\Big(\frac{\rho}{H}, \frac{r}{H}\Big)\rho\, d\rho = \theta_1 H[\delta - f(r)], \qquad 0 \le r \le a,$$

$$K(\sigma,\tau) = \int\limits_0^\infty L(u)J_0(\sigma u)J_0(\tau u)\, du, \qquad \theta_1 = \frac{G_1}{1-\nu_1}. \tag{4.1}$$

Here a is the radius of the contact region, δ is the translational displacement of the die along the axis orthogonal to the foundation surface, $f(r)$ is the function of the die-foundation shape, H is the thickness of the two-layer plate, and G_1 and ν_1 are the mechanical characteristics of the upper layer. The function $L(u)$ coincides with that obtained in the auxiliary problem (see [6]).

The integral equation (4.1) is supplemented by the condition of die equilibrium

$$P = 2\pi \int\limits_0^a q(\rho)\rho\, d\rho, \tag{4.2}$$

which serves to determine the relation between the pressing force P and the depth covered by the die δ. If the radius of the contact region a is not defined by the die shape in the problem formulation, it is prescribed by the condition

$$q(a) = 0. \tag{4.3}$$

5. Method of Solving the Axisymmetric Problem. The integral equation of the first kind (4.1) can be reduced to the following integral equation of the second kind with a difference kernel [6, 7]:

$$p(x) - \frac{1}{\pi H}\int\limits_{-a}^a p(\xi)M\Big(\frac{\xi - x}{H}\Big)\, d\xi = \theta_1 g(x), \qquad |x| \le a,$$

$$M(y) = \int\limits_0^\infty [1 - L(u)]\cos(uy)\, du; \tag{5.1}$$

the functions $p(x)$ and $g(x)$ being even and related to the functions $q(r)$ and $\delta(r) = \delta - f(r)$ as

$$q(r) = \frac{2}{\pi}\Big[\frac{p(a)}{\sqrt{a^2 - r^2}} - \int\limits_r^a \frac{p'(\xi)\, d\xi}{\sqrt{\xi^2 - r^2}}\Big], \qquad g(x) = \delta(0) + |x|\int\limits_0^{|x|} \frac{\delta'(\rho)\, d\rho}{\sqrt{x^2 - \rho^2}}. \tag{5.2}$$

We introduce the dimensionless complexes

$$x' = \frac{x}{a}, \quad \xi' = \frac{\xi}{a}, \quad \varphi(x') = \frac{p(ax')}{\theta_1 a}, \quad \psi(x') = \frac{g(ax')}{a}, \quad \lambda = \frac{H}{a}$$

and write the integral equation (5.1) as follows:

$$\varphi(x) - \frac{1}{2\pi\lambda} \int_{-1}^{1} \varphi(\xi) \Big[M\Big(\frac{\xi - x}{\lambda}\Big) + M\Big(\frac{\xi + x}{\lambda}\Big)\Big] \, d\xi = \psi(x), \qquad |x| \le 1 \tag{5.3}$$

(hereinafter, we omit the primes at x' and ξ' for simplicity).

To approximately solve the integral equation (5.3), we use the collocation method described in [8, 9]. We construct the following even Lagrange interpolation polynomial over the zeros of the Legendre polynomial for the function $\varphi(x)$:

$$P_{2N+1}(x) = \frac{1}{2^{2N+1}(2N+1)!} \, \frac{d^{2N+1}(x^2-1)^{2N+1}}{dx^{2N+1}}.$$

It has the form

$$\varphi(x) \approx \frac{\varphi(0) P_{2N+1}(x)}{x P'_{2N+1}(0)} + 2 \sum_{n=1}^{N} \frac{\varphi(x_n) x P_{2N+1}(x)}{(x^2 - x_n^2) P'_{2N+1}(x_n)}$$
$$[P_{2N+1}(x_n) = 0, \quad n = 0, 1, \ldots, N, \quad x_0 = 0]. \tag{5.4}$$

Note that it is possible to pass to polynomials of even powers because

$$\frac{x P_{2N+1}(x) - x_n P_{2N+1}(x_n)}{x^2 - x_n^2} = \sum_{i=0}^{N} a_{in} P_{2i}(x).$$

Then, Eq. (5.4) acquires the form

$$\varphi(x) \approx \sum_{i=0}^{N} P_{2i}(x) \Big[\frac{\varphi(0)}{P'_{2N+1}(0)} \, a_{i0} + 2 \sum_{n=1}^{N} \frac{\varphi(x_n)}{P'_{2N+1}(x_n)} \, a_{in} \Big]. \tag{5.5}$$

Using the property of orthogonality of the Legendre polynomials, we use Eq. (5.5) to obtain the following Gaussian-type quadrature formula:

$$\int_{-1}^{1} \varphi(\xi) \, d\xi \approx 2 \Big[\frac{\varphi(0)}{P'_{2N+1}(0)} \, a_{00} + 2 \sum_{n=1}^{N} \frac{\varphi(x_n)}{P'_{2N+1}(x_n)} \, a_{0n} \Big]. \tag{5.6}$$

Applying Eq. (5.6) for approximate calculation of the integral in Eq. (5.3) and assuming that $x = x_m$, where x_m are zeros of the polynomial P_{2N+1}, we obtain a system of linear algebraic equations with respect to $\varphi(x_m)$ ($m = 0, 1, \ldots, N$):

$$\varphi(x_m) - \frac{1}{\pi\lambda} \Big\{ \frac{\varphi(0) a_{00}}{P'_{2N+1}(0)} \Big[M\Big(-\frac{x_m}{\lambda}\Big) + M\Big(\frac{x_m}{\lambda}\Big) \Big]$$
$$+ 2 \sum_{n=1}^{N} \frac{\varphi(x_n)}{P'_{2N+1}(x_n)} \, a_{0n} \Big[M\Big(\frac{x_n - x_m}{\lambda}\Big) + M\Big(\frac{x_n + x_m}{\lambda}\Big) \Big] \Big\} = \psi(x_m).$$

Solving this system, we find the approximate expression for the function $p(x)$:

$$p(x) = \theta_1 a \sum_{i=0}^{N} a_i P_{2i}\Big(\frac{x}{a}\Big),$$
$$a_i = \frac{\varphi(0)}{P'_{2N+1}(0)} \, a_{i0} + 2 \sum_{n=1}^{N} \frac{\varphi(x_n)}{P'_{2N+1}(x_n)} \, a_{in}. \tag{5.7}$$

TABLE 2

λ	c_1	c_2	λ	c_1	c_2
1/4	5.227	2.298	2	3.363	1.429
1/2	4.568	1.927	4	3.010	1.353
1	3.926	1.624			

Substituting (5.7) into (5.2), we find the expression for the function $q(r)$:

$$q(r) = \frac{2\theta_1 a}{\pi} \sum_{i=0}^{N} a_i \Big[\frac{1}{\sqrt{a^2 - r^2}} - \frac{1}{a} \sum_{m=0}^{i-1} (-1)^{i-m-1} \frac{(4i-4m-1)(2i-2m-2)!!}{(2i-2m-1)!!} P_{2i-2m-1}\Big(\sqrt{1 - \frac{r^2}{a^2}}\Big)\Big]. \quad (5.8)$$

Relations (4.2) and (4.3), by virtue of (5.8), are written as

$$P = 4\theta_1 a^2 a_0, \qquad \sum_{i=0}^{N} a_i = 0.$$

As an example, we consider a die with a parabolic profile under the action of a centrally applied force. The function of the die-foundation shape has the form $f(r) = r^2/(2R)$, where R is the radius of curvature at the apex of the parabola. Using the values of the parameters

$$G_2 = 3G_1/2, \quad G_3 = 2G_1, \quad \nu_1 = 0.25, \quad \nu_2 = 0.35, \quad \nu_3 = 1/3, \quad H/h = 2,$$

we find the dependence of the coupling coefficients between $P/(\theta_1 a^2)$ and δ/a, and also between $P/(\theta_1 a^2)$ and a/R, on λ. The values of $c_1 = P/(\theta_1 a\delta)$ and $c_2 = PR/(\theta_1 a^3)$ calculated for different values of the parameter λ are listed in Table 2.

6. Auxiliary and Contact Problems for a Transversely Isotropic Layer. We consider an axisymmetric problem of equilibrium of a transversely isotropic elastic layer adherent to an elastic isotropic half-space. The axis of symmetry z is directed normal to the plane of isotropy. The layer thickness is denoted by H. The elastic constants for the isotropic half-space are E_2 and ν_2.

The boundary conditions of the problem have the following form:

— for $z = H$,

$$\sigma_z^{(1)} = -\tilde{q}(r), \qquad \tau_{rz}^{(1)} = 0,$$
$$\tilde{q}(r) = q(r), \quad 0 \le r \le a, \qquad \tilde{q}(r) = 0, \quad a < r < \infty; \quad (6.1)$$

— for $z = 0$,

$$u_1 = u_2, \quad w_1 = w_2, \quad \sigma_z^{(1)} = \sigma_z^{(2)}, \quad \tau_{rz}^{(1)} = \tau_{rz}^{(2)}. \quad (6.2)$$

Here u and w are the displacements along the axes of the cylindrical coordinate system r and z, respectively, σ_z is the normal stress, and τ_{rz} is the shear stress. The boundary conditions (6.1) and (6.2) should be supplemented by the absence of displacements in the structure for $r \to \infty$ and $z \to -\infty$.

For a transversely isotropic layer, we express strains via stresses [10]:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\phi) - \frac{\nu_1}{E_1}\sigma_z, \qquad \varepsilon_\phi = \frac{1}{E}(\sigma_\phi - \nu\sigma_r) - \frac{\nu_1}{E_1}\sigma_z,$$
$$\varepsilon_z = \frac{\nu_1}{E_1}\sigma_z - \frac{\nu_1}{E_1}(\sigma_r + \sigma_\phi), \qquad \gamma_{rz} = \frac{1}{G_1}\tau_{rz}. \quad (6.3)$$

Here $\varepsilon_r = \partial u/\partial r$, $\varepsilon_\phi = u/r$, $\varepsilon_z = \partial w/\partial z$, and $\gamma_{rz} = \partial u/\partial z + \partial w/\partial r$. Solving this system with respect to stresses, we obtain

$$\sigma_r = a_1\varepsilon_r + a_2\varepsilon_\phi + a_3\varepsilon_z,$$
$$\sigma_\phi = a_2\varepsilon_r + a_1\varepsilon_\phi + a_3\varepsilon_z, \quad (6.4)$$
$$\sigma_z = a_3\varepsilon_r + a_3\varepsilon_\phi + a_4\varepsilon_z,$$

where $a_1 = E(E_1 - \nu_1^2 E)/D$, $a_2 = E(\nu E_1 + \nu_1^2 E)/D$, $a_3 = EE_1\nu_1(1+\nu)/D$, $a_4 = E_1^2(1-\nu^2)/D$, and $D = (1-\nu^2)E_1 - 2E\nu_1^2(1+\nu)$. Further we assume that $D > 0$.

We substitute the resultant expressions for stresses into the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\phi}{r} = 0, \qquad \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0$$

and finally we obtain the Lamé equations

$$\begin{gathered} a_1 \tilde{L}^2 u + G_1 \frac{\partial^2 u}{\partial z^2} + (a_3 + G_1) \frac{\partial^2 w}{\partial r\, \partial z} = 0, \\ G_1 L^2 w + a_4 \frac{\partial^2 w}{\partial z^2} + (a_3 + G_1) \frac{\partial}{\partial z} \hat{L} u = 0. \end{gathered} \tag{6.5}$$

In these equations, we use the following notation for the differential operators:

$$L^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \qquad \hat{L} = \frac{\partial}{\partial r} + \frac{1}{r}, \qquad \tilde{L}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}.$$

We introduce the displacement function χ: $u = -(a_3 + G_1)\, \partial^2 \chi / \partial r\, \partial z$. We express w in terms of χ from the first equation in (6.5). We have $w = (a_1 L^2 + G_1\, \partial^2/\partial z^2)\chi$. We substitute this expression into the second equation in (6.5) and transform it. As a result, we obtain the equation in fourth-order partial derivatives for the function χ:

$$\begin{gathered} \Big(\frac{\partial^4}{\partial z^4} + 2A \frac{\partial^2}{\partial z^2} L^2 + BL^4\Big)\chi = 0, \\ 2A = \frac{a_4 a_1 - 2a_3 G_1 - a_3^2}{a_4 G_1}, \qquad B = \frac{a_1}{a_4}. \end{gathered} \tag{6.6}$$

Applying Hankel's transform

$$\chi = \int_0^\infty X(\gamma, z) J_0(\gamma z) \gamma\, d\gamma,$$

where J_0 is the Bessel function, we use Eq. (6.6) to obtain the fourth-order ordinary differential equation. Its solution in the general form is described by the equation

$$X_1 = c_1\, e^{k_1 \gamma z} + c_2\, e^{k_2 \gamma z} + c_3\, e^{k_3 \gamma z} + c_4\, e^{k_4 \gamma z}.$$

The coefficients k_1, k_2, k_3, and k_4 are the roots of the equation $k^4 - 2Ak^2 + B = 0$. For the underlying half-space, we obtain the expression of a similar transform:

$$X_2 = (d_1 + \gamma z d_2)\, e^{\gamma z}.$$

Six functions c_1, c_2, c_3, c_4, d_1, and d_2 of the parameter γ should be found from the boundary conditions (6.1) and (6.2), for which purpose we express these boundary conditions [with the help of Eqs. (6.3) and (6.4)] in terms of the functions χ_j $(j = 1, 2)$. After that, we present the discontinuous function $\tilde{q}(r)$ in the form of Hankel's integral

$$\tilde{q}(r) = \int_0^\infty Q(\gamma) J_0(\gamma r) \gamma\, d\gamma$$

and write all boundary conditions in transforms. Using the expressions for transforms, we obtain a system of six algebraic equations for determining the sought functions c_1, c_2, c_3, c_4, d_1, and d_2. Solving this system, we obtain the expression

$$w(r, H) = \frac{1}{\theta_1} \int_0^\infty L(\gamma H) Q(\gamma) J_0(\gamma r)\, d\gamma$$

necessary to formulate the contact problem.

Note that the function $L(u)$ whose expression is not given here has the same meaning as in the previous problem and possesses similar asymptotic properties:

$$\begin{aligned} L(u) &= 1 + O(e^{-2uk_1}), \qquad && u \to \infty, \\ L(u) &= n + O(u), && u \to 0. \end{aligned}$$

TABLE 3

λ	c_1	c_2	λ	c_1	c_2
1/4	3.122	1.463	2	2.754	1.336
1/2	2.965	1.392	4	2.710	1.334
1	2.836	1.351			

The resultant function $L(u)$ determining the character of the kernel in the integral equation (4.1) allows us to use the results of Sec. 4.

As an example, we consider a die with a parabolic profile under the action of a centrally applied force. The function of the die-foundation shape has the form $f(r) = r^2/(2R)$, where R is the radius of curvature at the apex of the parabola. Using the values of the parameters

$$E_1 = 0.915E, \quad G_1 = 0.382E, \quad E_2 = 1.281E, \quad \nu = 0.22, \quad \nu_1 = 0.24, \quad \nu_2 = 0.28,$$

we find the dependence of the coupling coefficients between $P/(\theta_1 a^2)$ and δ/a, and also $P/(\theta_1 a^2)$ and a/R, on λ. The values of $c_1 = P/(\theta_1 a\delta)$ and $c_2 = PR/(\theta_1 a^3)$ calculated for different values of the parameter λ are listed in Table 3.

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